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Parametrically excited solitary waves

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A modulated cross-wave of resonant frequency ω_1, carrier frequency $\omega = \omega_1\{1+O(\epsilon)\}$, slowly varying complex amplitude $O(\epsilon^{\frac{1}{2}}b)$, longitudinal scale $b/\epsilon^{\frac{1}{2}}$ and timescale $1/\epsilon\omega$ is induced in a long channel of breadth b that contains water of depth d and is subjected to a vertical oscillation of amplitude $O(\epsilon b)$ and frequency 2ω, where $0 < \epsilon \ll 1$. The complex amplitude satisfies a cubic Schrödinger equation, generalized to incorporate weak damping and the parametric excitation. A solution is obtained that describes the standing solitary wave observed by Wu, Keolian & Rudnick (1984). The results depend on both d/b and l_*/b, where l_* is the capillary length ($l_* = 2.7$ mm for clean water), and solitary waves are impossible if $d/b < 0.325$ for $l_*/b = 0$ or if $l_*/b > 0.045$ for $d/b \gtrsim 1$. The corresponding cnoidal waves (of which the solitary wave is a limiting case) are considered in an appendix.

1. Introduction

The following investigation was stimulated by Wu, Keolian & Rudnick's (1984) observation of a standing solitary wave in a long channel subjected to either a vertical oscillation at a frequency of approximately twice the natural frequency of the dominant cross-wave or a lateral oscillation at a frequency approximately equal to the natural frequency. This standing wave appears to be related to the trapped cross-waves that sometimes appear in front of a wavemaker at either the frequency of the wavemaker (Barnard, Mahony & Pritchard 1977) or half that frequency (Barnard & Pritchard 1972; Jones 1984). The amplitudes reported by Wu *et al.* were rather large ($\sim$ 2 cm), and the corresponding lengths were of the order of the channel width, but it appears that waves of smaller amplitude should be correspondingly longer and may be regarded as slowly modulated, nonlinear cross-waves. I consider here the case of parametric (vertical) excitation.

I was first led to reinvestigate the problem of Faraday resonance, for which standing waves are parametrically excited in a basin that is subjected to a vertical oscillation at approximately twice the natural frequency of a particular mode. The central result of this investigation (Miles 1984) is that if (i) the free-surface displacement is represented by the expansion

$$\eta(\boldsymbol{x}, t) = \eta_n(t)\,\psi_n(\boldsymbol{x}), \tag{1.1}$$

where the $\psi_n(\boldsymbol{x})$ constitute a complete set of normal modes with r.m.s. values of unity and the $\eta_n(t)$ are the corresponding generalized coordinates, and (ii) the driving frequency 2ω approximates twice that of a particular (typically, but not necessarily, the dominant) mode, say $n = 1$, for which (by hypothesis)

$$\eta_1 = a[p(\tau)\cos\omega t + q(\tau)\sin\omega t], \quad \tau = \epsilon\omega t, \tag{1.2a, b}$$

where a is a reference amplitude and ϵ is a small parameter, then (iii) the dimensionless amplitudes p and q are governed by the nonlinear evolution equations

$$\dot{p}+\alpha p+[\beta-\gamma+A(p^2+q^2)]\,q = 0, \tag{1.3a}$$

$$\dot{q}+\alpha q-[\beta+\gamma+A(p^2+q^2)]\,p = 0, \tag{1.3b}$$

where α, β and γ are measures of damping, frequency offset, and driving amplitude that are $O(1)$ as $\epsilon\to 0$ (see §2), and A is a measure of nonlinearity that relates the frequency of free ($\alpha=\gamma=0$) oscillations of small but finite amplitude to the natural frequency of the mode under consideration. The appropriate normal modes in the present problem are the two-dimensional cross-waves, for which $\psi_n = \surd 2\cos(n\pi y/b)$ in a channel with walls at $y = 0, b$; and the corresponding value of A, which depends only on the depth ratio d/b, has been calculated by Tadjbakhsh & Keller (1960).

The two-dimensional formulation may be modified to incorporate a weak spatial variation (see §3), for which p and q are functions of both the slow time τ and the stretched variable $X = O(\epsilon^{\frac{1}{2}}x/b)$, where x is measured along the channel. This leads to the introduction of the spatial derivatives q_{XX} and p_{XX} in (1.3a) and (1.3b) respectively, after which the two evolution equations may be combined to obtain a generalized (to incorporate damping and parametric excitation), cubic Schrödinger equation (§4), which admits a standing solitary-wave solution. (This solitary wave is a limiting case of a cnoidal wave, which I examine briefly in Appendix A.)

On first carrying out the calculation outlined above, I had incorporated the constraint $\eta_0 = 0$, where η_0, the mean displacement, is the coefficient of $\psi_0 = 1$ in the Fourier expansion (1.1). This constraint is manifestly necessary for conservation of mass in the two-dimensional problem, in which η is independent of x, but the argument is no longer compelling if the integral $\iint \eta\,dx\,dy$ has a finite value (as it does for the solitary wave), which may be compensated by an infinitesimal change in the mean elevation over the infinitely long channel. The paradox is resolved by the requirement that the velocity potential, which comprises a spatially independent component that is linear in t in the two-dimensional problem (wherein it contributes to the perturbation pressure through the term $\partial\phi/\partial t$ in Bernoulli's equation but does not contribute to the kinetic energy), have bounded x- (or X-) derivatives as $t\uparrow\infty$ in the three-dimensional problem.

The results obtained in §§4 and 5 have been obtained independently, and through a rather different procedure, by Larraza & Putterman (1984) for free waves ($\alpha=\gamma=0$ herein). On comparing our results, we found a small discrepancy in the parameter A. I had originally used Tadjbakhsh & Keller's result for the two-dimensional problem, (2.11) below, and it was only after learning of Larraza & Putterman's result that I developed the argument outlined in the preceding paragraph and obtained the result (3.17) below for A. To the extent that our results overlap, then, priority is due to them.

It is evident from the relatively small dimensions of the channel used by Wu *et al.* that capillary effects must have been significant in their experiments. These effects, which are examined in Appendix B, modify the parameters in the evolution equations (but not the form of these equations) and render solitary waves impossible if the surface tension exceeds a certain critical value.

2. Resonantly forced cross-waves

We consider the two-dimensional cross-waves induced in a long rectangular channel of breadth b and depth d that is subjected to the vertical oscillation

$$z_0 = a_0 \cos 2\omega t \quad \left(0 < \frac{\omega^2 a_0}{g} \ll 1\right), \tag{2.1}$$

where ω approximates the natural frequency

$$\omega_1 = (gk \tanh kd)^{\frac{1}{2}} \quad \left(k = \frac{\pi}{b}\right) \tag{2.2}$$

of the dominant mode. The free-surface displacement (relative to the plane of the level surface, which moves with the channel) admits the Fourier expansion

$$\eta = \eta_n(t)\,\psi_n(y), \quad \psi_n = \surd 2 \cos nky, \tag{2.3a,b}$$

where the η_n are generalized coordinates and the repeated index n is summed from 1 to ∞; $\eta_0 = 0$ by virtue of conservation of mass (but see §3). The assumptions

$$\beta \equiv \frac{\omega^2 - \omega_1^2}{2\epsilon\omega_1^2} = O(1), \quad \gamma \equiv \frac{\omega^2 a_0}{\epsilon g} = O(1), \tag{2.4a,b}$$

where ϵ is a small, positive scaling parameter (ϵ is determined by $\gamma \equiv 1$ in M84 (which, here and subsequently, signifies Miles 1984) but is ultimately defined by (2.7) in the present formulation), permit the η_n to be posited in the form (M84(3.1))

$$\eta_n = \delta_{1n}\, a[p(\tau)\cos\theta + q(\tau)\sin\theta] + a^2 k \tanh kd[A_n(\tau)\cos 2\theta + B_n(\tau)\sin 2\theta + C_n(\tau)], \tag{2.5}$$

where $a = O(\epsilon^{\frac{1}{2}}/k)$ is a lengthscale ($a = l$ in M84(3.1)), p, q, A_n, B_n and C_n are slowly varying amplitudes, and

$$\theta = \omega t, \quad \tau = \epsilon\omega t \tag{2.6a, b}$$

are fast and slow dimensionless times.

The Lagrangian of the motion may be constructed as in M84 §3 and averaged over θ, after which the A_n, B_n and C_n (which are significant only for $n = 0$ and 2 in the present problem by virtue of the orthogonality of ψ_n and ψ_1^2 for $n \neq 0$ or 2) may be determined as functions of p and q. The end result, as given by M84(3.9) and (3.10) after choosing

$$a = 2\epsilon^{\frac{1}{2}}\lambda \quad (\lambda = k^{-1}\tanh kd) \tag{2.7}$$

and allowing for the difference in scaling (a versus l), is†

$$\langle L\rangle = \epsilon g a^2\{\tfrac{1}{2}(\dot{p}q - p\dot{q}) + H(p,q)\}\{1 + O(\epsilon)\}, \tag{2.8}$$

where $\langle\ \rangle$ signifies a joint average over y and θ, the dots imply differentiation with respect to τ, and

$$H = \tfrac{1}{2}\beta(p^2+q^2) + \tfrac{1}{2}\gamma(p^2-q^2) + \tfrac{1}{4}A_0(p^2+q^2)^2. \tag{2.9}$$

The parameter A_0, which relates the frequency of free ($\gamma = 0$) oscillations of small

† The true Lagrangian for a unit length of the channel is $\rho b L$, where ρ is the fluid density.

but finite amplitude in the dominant mode to the natural frequency according to (M84(3.11))

$$\left(\frac{\omega}{\omega_1}\right)^2 = 1 - A_0\left(\frac{\langle \eta_1^2 \rangle}{\lambda^2}\right), \tag{2.10}$$

is given by (Tadjbakhsh & Keller 1960)

$$A_0 = \tfrac{1}{8}(2T^4 + 3T^2 + 12 - 9T^{-2}), \quad T \equiv \tanh kd. \tag{2.11a, b}$$

3. Spatial modulation

We now suppose that the motion varies weakly in the x (longitudinal) direction, with a lengthscale that is $O(1/\epsilon^{\frac{1}{2}}k)$, and regard p and q as functions of both the slow time τ and the stretched variable

$$X = 2(\epsilon T)^{\frac{1}{2}} kx. \tag{3.1}$$

The Lagrangian density in x, y, t is given by (Miles 1977, after replacing y and g therein by z and $g + \ddot{z}_0$)

$$L = \xi\eta_t - \frac{1}{2}\int_{-d}^{\eta} (\nabla\phi)^2 \,\mathrm{d}z - \tfrac{1}{2}(g + \ddot{z}_0)\,\eta^2, \tag{3.2}$$

where ξ is the velocity potential at $z = \eta$ (ξ and η are canonically conjugate variables in Hamilton's sense). Both ξ and η admit Fourier expansions of the form (2.3a); however, we can no longer impose the *a priori* restriction $\eta_0 = 0$, since a finite value of $\iint \eta_0 \,\mathrm{d}x\,\mathrm{d}y$ may be compensated by an infinitesimal change in the mean elevation over the infinite surface to ensure conservation of mass. We therefore posit the joint expansion

$$(\xi, \eta) = (\xi_n, \eta_n)\,\psi_n(y), \quad \psi_n = (2 - \delta_{0n})^{\frac{1}{2}} \cos nky, \tag{3.3a, b}$$

where repeated indices now are summed from 0 to ∞, δ_{0n} is the Kronecker delta, and ξ_n and η_n are function of θ, τ and X. Substituting (3.3) into (3.2), averaging over y (we denote this average by square brackets), evaluating the kinetic-energy integral as in §4 of M76 (Miles 1976), and invoking $\phi_x^2 = O(\epsilon k^2 \xi_1^2)$, we obtain

$$[L] = \xi_n \eta_{nt} - \tfrac{1}{2}\mathscr{k}_{mn}\xi_m\xi_n - \frac{1}{2b}\int_0^b \mathrm{d}y \int_{-d}^{\eta} \phi_x^2 \,\mathrm{d}z - \tfrac{1}{2}(g + \ddot{z}_0)\,\eta_n\eta_n \tag{3.4a}$$

$$\equiv [L_2] + \Delta[L], \tag{3.4b}$$

wherein an error factor of $1 + O(\epsilon)$ is implicit,

$$\mathscr{k}_{mn} = \delta_{mn}\mathscr{k}_n + \mathscr{k}_{lmn}\eta_l + \tfrac{1}{2}\mathscr{k}_{jlmn}\eta_j\eta_l + \dots \tag{3.5}$$

is given by M76(4.4),

$$\mathscr{k}_n = nk \tanh nkd, \tag{3.6}$$

$[L_2]$ is the reduced form of $[L]$ for $\eta_0 = \phi_x = 0$ and therefore must be equivalent to $[L]$ in §2, and $\Delta[L]$ comprises the incremental terms in η_0 and ϕ_x.

It suffices for the calculation of $\Delta[L]$ to use the linear approximation to the solution of the kinematic boundary-value problem

$$\nabla^2\phi = 0, \quad \phi_y = 0 \quad \left(y = 0, \frac{\pi}{k}\right), \quad \phi_z = 0 \quad (z = -d), \quad \phi_z = \eta_t \quad (z = \eta), \tag{3.7a, b, c, d}$$

which is given by

$$\phi = (k \sinh kd)^{-1} \eta_{1t} \psi_1(y) \cosh k(z+d) \{1 + O(\epsilon^{\frac{1}{2}})\}. \tag{3.8}$$

(Note that the error factor $1 + O(\epsilon^{\frac{1}{2}})$ in ϕ ultimately implies an error factor of $1 + O(\epsilon)$ in L by virtue of the requirement (Hamilton's principle) that L be stationary with respect to first-order variations about the exact solution.) Substituting (3.8) into the integral in (3.4a), combining the result with the remaining incremental terms, $\xi_0 \eta_{0t}, \frac{1}{2}k_{011} \eta_0 \xi_1^2$ (the remaining contributions of η_0 to $\frac{1}{2}k_{mn} \xi_m \xi_n$ are negligible in the present approximation), and $\frac{1}{2}(g+\ddot{z}_0)\eta_0^2 \approx \frac{1}{2}g\eta_0^2$, invoking $k_{011} = k^2(1-T^2)$, and approximating ξ_1 by η_{1t}/kT, which follows from (3.8), we obtain

$$\Delta[L] = \xi_0 \eta_{0t} - \tfrac{1}{2}(T^{-2}-1)\, \eta_0 \eta_{1t}^2 - \tfrac{1}{2}g\eta_0^2 - \frac{1}{4}\frac{B}{k^3 T^2} \eta_{1xt}^2, \tag{3.9}$$

where

$$B = \tanh kd + kd \operatorname{sech}^2 kd = T + kd(1-T^2). \tag{3.10}$$

We now invoke the requirement that L be stationary with respect to variations of each of ξ_0 and η_0 to obtain

$$\eta_{0t} = 0, \quad \xi_{0t} = -\tfrac{1}{2}(T^{-2}-1)\, \eta_{1t}^2 - g\eta_0. \tag{3.11a, b}$$

The implicit solution of (3.11) adopted in §2 (as demanded by conservation of mass) is

$$\eta_0 = 0, \quad \xi_0 = -\tfrac{1}{2}(T^{-2}-1) \int \eta_{1t}^2 \, \mathrm{d}t. \tag{3.12a, b}$$

This implies that ξ_0 contains a term that is linear in t; however, this term is independent of x and enters the solution only through the contribution of $-\rho\phi_t$ to the perturbation pressure (cf. M76, §5). If, on the other hand, η_0 and η_1 depend on x, the resulting divergence of ξ_0 with t is unacceptable and must be prevented by choosing

$$g\eta_0 = -\tfrac{1}{2}(T^{-2}-1)\, \langle \eta_{1t}^2 \rangle, \tag{3.13}$$

which renders ξ_0 oscillatory in t (with frequency 2ω). Substituting η_1 from (2.5) into (3.13), averaging, and invoking $\omega^2 \approx \omega_1^2 = gkT$ and (2.7) for a, we obtain

$$\eta_0 = -\epsilon\lambda(1-T^2)\,(p^2+q^2). \tag{3.14}$$

Returning to (3.9), invoking (3.1), (3.11a), (3.13) and (2.5), averaging over θ, and invoking $\omega^2 \approx \omega_1^2$, we obtain

$$\langle \Delta L \rangle = \tfrac{1}{8}g^{-1}(T^{-2}-1)^2 \langle \eta_{1t}^2 \rangle^2 - \epsilon \frac{B}{kT} \langle \eta_{1Xt}^2 \rangle \tag{3.15a}$$

$$= \epsilon g a^2 \{\tfrac{1}{8}(1-T^2)^2\,(p^2+q^2)^2 - \tfrac{1}{2}B(p_X^2 + q_X^2)\}. \tag{3.15b}$$

It then follows, by adding (3.15b) to (2.8), that longitudinal modulation may be incorporated in the formulation of §2 by replacing (2.9) by

$$H = \tfrac{1}{2}\beta(p^2+q^2) + \tfrac{1}{2}\gamma(p^2-q^2) + \tfrac{1}{4}A(p^2+q^2)^2 - \tfrac{1}{2}B(p_X^2+q_X^2), \tag{3.16}$$

where

$$A = A_0 + \tfrac{1}{2}(1-T^2)^2 = \tfrac{1}{8}(6T^4 - 5T^2 + 16 - 9T^{-2}). \tag{3.17}$$

We remark that A, which is plotted in figure 1, is a monotonically increasing function of kd that is asymptotic to 1 as $kd \uparrow \infty$; it vanishes at $kd = 1.022$ (cf. 1.058 for A_0).

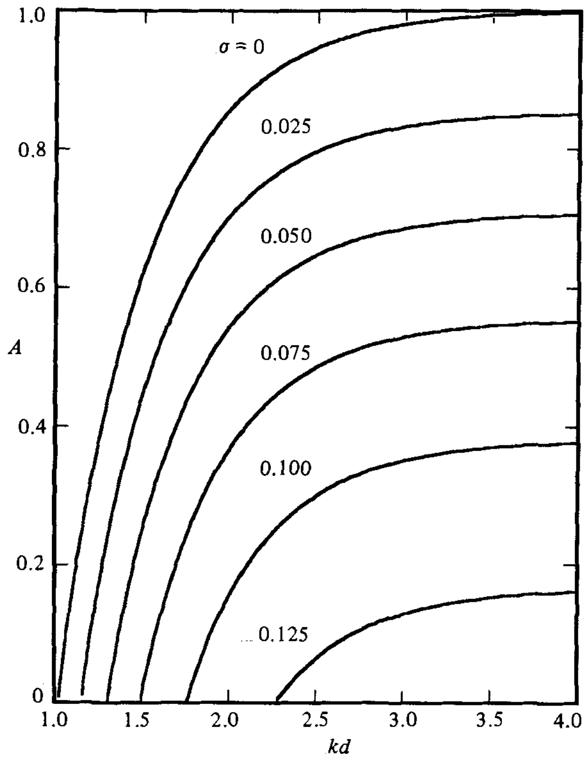

FIGURE 1. $A \equiv A_*$, as given by (B 4) with $\sigma = k^2 l_*^2$ as the family parameter. The result for $\sigma = 0$ corresponds to (3.17). $A < 0$ for all kd if $0.14 < \sigma < 0.25$.

(It is worth noting that A may be formally calculated from M84(E7) by including $m = 0$ in the summation therein and invoking the limit $a_m = \infty$ as $m \to 0$; excluding $m = 0$ in this result yields (2.11*a*). Both A and B are modified by capillary effects (Appendix B).

4. Evolution equations

It follows from the application of Hamilton's principle to (2.8) that p and q are canonical variables, for which the evolution equations are (cf. (1.3))

$$-p_\tau = \frac{\delta H}{\delta q} = Bq_{XX} + [\beta - \gamma + A(p^2+q^2)]\,q, \tag{4.1a}$$

$$q_\tau = \frac{\delta H}{\delta p} = Bp_{XX} + [\beta + \gamma + A(p^2+q^2)]\,p. \tag{4.1b}$$

Weak, linear damping may be incorporated at this stage by replacing ∂_τ by $\partial_\tau + \alpha$ in (4.1), where

$$\alpha = \delta/\epsilon, \tag{4.2}$$

and δ is the ratio of actual to critical damping for free oscillations (of sufficiently small amplitude) in the resonant mode and is assumed to be $O(\epsilon)$ in the present context (damping dominates the effects being considered here if $\delta \gg \epsilon$). The damping ratio is perhaps best determined by measuring the decay of the free cross-wave; however, it may be calculated for a channel with hydrophilic walls (Appendix C).

Introducing the complex variable†

$$r = p + iq \tag{4.3}$$

and combining (4.1 *a*, *b*), augmented by the aforementioned damping terms, we obtain

$$i(r_\tau + \alpha r) + Br_{XX} + (\beta + A|r|^2)\,r + \gamma r^* = 0, \tag{4.4}$$

where $r^* \equiv p - iq$ is the complex conjugate of r. We remark that

$$\eta_1 + \frac{i\omega}{g}\xi_1 = ar(X,\tau)\,e^{-i\omega t}. \tag{4.5}$$

5. Solitary waves

If $\alpha = \gamma = 0$ (free, undamped motion), (4.4) reduces to the cubic Schrödinger equation (Whitham 1974), which admits solitary-wave solutions if $A > 0$, in particular the standing wave

$$r = e^{i(\beta + \frac{1}{2}A)\tau}\,\mathrm{sech}\left[\left(\frac{A}{2B}\right)^{\frac{1}{2}} X\right] \quad (\alpha = \gamma = 0,\ A > 0). \tag{5.1}$$

Substituting (5.1) into (4.5) and invoking (3.1) for X and (2.7) for ϵ, we obtain

$$\eta_1 = \mathrm{Re}\left\{a\,e^{-i\omega t}\,\mathrm{sech}\frac{x}{l}\right\}, \quad l = \frac{(2BT/A)^{\frac{1}{2}}}{k^2 a}, \tag{5.2\textit{a}, \textit{b}}$$

within $1 + O(\epsilon^{\frac{1}{2}})$, where ω now is given by (cf. (2.10))

$$\omega = \omega_1\left[1 - \tfrac{1}{8}A\left(\frac{a}{\lambda}\right)^2\right]. \tag{5.3}$$

Very weak damping ($0 < \alpha \ll 1$) may be incorporated by regarding the parameters in the solution as slowly varying functions of $\alpha\tau$ (Lamb 1980).

Returning to (4.4), we seek a steady solution in the form, suggested by (5.1),

$$r = e^{i\psi}\,\mathrm{sech}\left[\left(\frac{A}{2B}\right)^{\frac{1}{2}} X\right], \tag{5.4}$$

where ψ is a phase constant. Substituting (5.4) into (4.4), we obtain

$$\left.\begin{matrix}\psi \\ \frac{1}{2}\pi - \psi\end{matrix}\right\} = \tfrac{1}{2}\sin^{-1}\frac{\alpha}{\gamma}, \quad \beta \pm (\gamma^2 - \alpha^2)^{\frac{1}{2}} + \tfrac{1}{2}A = 0. \tag{5.5\textit{a}, \textit{b}}$$

An analysis based on evolution equations obtained by positing $r = R(\tau)$ sech $[(A/2B)^{\frac{1}{2}}X]$ in $\langle L\rangle$ and integrating over X implies that only the lower choice in (5.5) yields a stable solution. Invoking (2.4*a*, *b*), (2.7) and (4.2), we then obtain

$$a^2 = \frac{4}{A}\lambda^2\left\{1 - \left(\frac{\omega}{\omega_1}\right)^2 + 2\left[\left(\frac{\omega^2 a_0}{g}\right)^2 - \delta^2\right]^{\frac{1}{2}}\right\}. \tag{5.6}$$

It follows that necessary conditions for the existence of the standing wave (5.4), in addition to $A > 0$ ($d/b > 0.325$), are (within $1 + O(\epsilon)$)

$$a_0 > \frac{g\delta}{\omega^2} \approx \frac{\delta}{kT} \tag{5.7\textit{a}}$$

and

$$\omega^2 < \omega_1^2\{1 + 2[(ka_0 T)^2 - \delta^2]^{\frac{1}{2}}\}. \tag{5.7\textit{b}}$$

† A slightly rescaled form of (4.4) may be derived through the canonical transformation $w = (p + iq)/\sqrt{2}$ and $w^* = (p - iq)/\sqrt{2}$, which implies $w_\tau = i(\delta H/\delta w_*)$.

The data for a typical run in the channel of Wu *et al.* are $b = 2.54$ cm, $d = 2.00$ cm, $g = 980$ cm/s, $a_0 = 0.075$ cm, $a = 1.7$ cm and $\omega = 32.7$ rad/s ($2\nu = 10.4$ Hz). The corresponding value of ω_1 is 34.6 rad/s (uncorrected for surface tension). The estimated damping ratio (see Appendix C) is $\delta = 0.015$, which is much smaller than $\omega^2 a_0/g = 0.082$, so that (5.7*a*) is amply satisfied. The corresponding critical value of ω/ω_1, as calculated from (5.7*b*), is 1.077; however, the incorporation of a surface tension of 72 dyn/cm (Appendix B, $\sigma \approx 0.112$) yields a critical value of 1.128, which exceeds the actual value of 0.946, as required. The calculated value of a, as given by (5.6) after incorporating surface tension and invoking (B 4) for the surface-tension-corrected value of A, is 2.3 cm, which, in view of the assumption of small amplitudes in the theoretical calculation, is perhaps closer than might have been expected to the observed value of 1.7 cm. The characteristic length l, as given by (5.2*b*) using (B 4) and (B 5) for A and B, is 1.1 cm, which compares with an observed value of roughly 1.4 cm (obtained by rescaling the observed value of 1.12 cm for an amplitude of 2.1 cm). The product la is 2.41 cm², which agrees (within the accuracy of the data) with the observed value of 2.35 cm².

Periodic (in X) solutions of (4.4) are considered in Appendix A.

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Appendix A. Cnoidal waves

The solitary wave (5.4) is a limiting case of the cnoidal wave

$$r = e^{i\psi} \operatorname{cn}\left[\kappa^{-1}\left(\frac{A}{2B}\right)^{\frac{1}{2}} X, \kappa\right] \quad (\kappa^2 < 1), \tag{A 1}$$

where ψ is given by (5.5*a*), cn is an elliptic cosine of modulus κ (the family parameter), (5.5*b*) is replaced by

$$\beta \pm (\gamma^2 - \alpha^2)^{\frac{1}{2}} + A(1 - \tfrac{1}{2}\kappa^{-2}) = 0, \tag{A 2}$$

and the right-hand side of (5.6) must be divided by $2 - \kappa^{-2}$. The assumptions (2.3) require $\kappa^{-2} = O(1)$. The conditions $A > 0$ and (5.7*a*) are unchanged; (5.7*b*) remains unchanged if $\kappa^2 > \frac{1}{2}$ but otherwise must be replaced by

$$\omega^2 > \omega_1^2\{1 + 2[(ka_0 T)^2 - \delta^2]^{\frac{1}{2}}\} \quad (\kappa^2 < \tfrac{1}{2}). \tag{A 3}$$

The solitary wave is recovered in the limit $\kappa \uparrow 1$.

The cnoidal wave (A 1) is (within a uniform translation of X) the most general steady solution of (4.4) with uniform phase. The presence of the complex-conjugate term γr^* prevents the introduction of an exponential (variable-phase) factor such as that which typically accompanies the solution of the cubic Schrödinger equation.

Appendix B. Capillary effects

The potential energy per unit area of free surface due to a surface tension ρT_1 is

$$V = \tfrac{1}{2}\rho T_1(\eta_x^2 + \eta_y^2). \tag{B 1}$$

Combining (2.3) and (2.5) in (B 1), regarding p, q, A_n, B_n and C_n as functions of X and τ, averaging over y and θ, invoking (2.7) and (3.1), and neglecting $O(\epsilon^2)$ relative to 1, we obtain

$$\langle V\rangle/\rho = T_1\, k^2a^2[\tfrac{1}{4}(p^2+q^2)+\epsilon T(p_X^2+q_X^2)+4\epsilon T^4(A_2^2+B_2^2+2C_2^2)], \tag{B 2}$$

which must be subtracted from (2.8). Proceeding as in M84, Appendix E, we find that (2.4*a*), (3.17) and (3.10) must be replaced by

$$\beta_* = (2\epsilon)^{-1}[(\omega/\omega_1)^2-1-\sigma], \tag{B 3}$$

$$A_* = \tfrac{1}{2}[1+(1-T^2)^2+\tfrac{1}{2}(1+4\sigma)^{-1}(1+T^2)^2-\tfrac{1}{4}(T^2-4\sigma)^{-1}(3-T^2)^2] \tag{B 4}$$

and

$$B_* = T+kd(1-T^2)+2\sigma T, \tag{B 5}$$

where

$$\sigma = T_1k^2/g \equiv k^2l_*^2, \tag{B 6}$$

and $l_* = (T_1/g)^{\frac{1}{2}}$ is the capillary length ($l_* = 2.7$ mm for clean water). A_* is plotted in figure 1.

The limiting values of A_* and B_* for deep water (note that kd typically is large in those parametric régimes for which capillary effects are significant) are

$$A_* \to \frac{1-6\sigma-8\sigma^2}{1-16\sigma^2}, \quad B_*\to 1+2\sigma \quad (T\uparrow 1). \tag{B 7a, b}$$

A_* vanishes for $\sigma = 0.14$, and solitary waves are impossible for $0.14 < \sigma < 0.25$.

It should be emphasized that the present formulation is valid only if β_* and $T^2-4\sigma$ are $O(1)$ as $\epsilon\downarrow 0$. $T^2-4\sigma = O(\epsilon)$ corresponds to internal resonance between the first and second modes, which then must be regarded as of comparable magnitude in the perturbation expansion.

Appendic C. Damping

The contribution of the boundary layers on the walls and bottom of the tank to the damping ratio δ are (Miles 1967, §7, wherein $\alpha = 2\pi\delta$, $L = b$, $B\to\infty$)

$$\delta_W = \frac{\hat{\epsilon}}{2\pi}\left[1+\frac{\pi-2kd}{\sinh 2kd}\right], \tag{C 1}$$

while that of the boundary layer at the free surface is

$$\delta_S = \tfrac{1}{4}\hat{\epsilon}(C_r-C_i)\coth kd, \tag{C 2}$$

where

$$\hat{\epsilon} = (2\nu/\omega)^{\frac{1}{2}}k, \tag{C 3}$$

and $C_r - C_i$, the ratio of the surface-film damping to that which would be produced by an inextensible film, may be approximated by unity in most applications (its maximum value is 2). If $kd \gtrsim 2$, the second term in the square brackets (which represents bottom damping) may be neglected in (C 1), and $\coth kd$ may be approximated by unity in (C 2). Combining these approximations, we obtain

$$\delta_W+\delta_S = 0.41\hat{\epsilon}. \tag{C 4}$$

The contribution of capillary hysteresis at the meniscus may be approximated by

$$\delta_L = \frac{8\kappa}{\pi}\frac{T_1}{a_1 bg}, \tag{C 5}$$

where κ is a semiempirical coefficient that may be approximated by 0.05 for a hydrophilic basin (e.g. clean water on glass) or 0.3 for a hydrophobic basin (e.g. clean water on lucite or brass), and a_1 is the amplitude of the oscillation at the meniscus ($a_1 = \sqrt{2}\, a$ in the notation of §2 above).

Substituting $a_1 = 2$ cm, $b = 2.54$ cm, $d = 2$ cm, $g = 980$ cm/s^2, $\nu = 0.01$ cm^2/s, $T_1 = 72$ dyn/cm and $\omega = 10\pi$ rad/s, the data for the apparatus of Wu *et al.*, into (C 3)–(C 5), and assuming $\kappa = 0.05$, we obtain $\delta_W + \delta_S = 0.013$ and $\delta_L = 0.002$.

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